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Block Toeplitz Matrices and the Two-Dimensional Coulomb Gas Near a Wall

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For the two-dimensional Coulomb gas on a lattice, at the special value of the dimensionless coupling constant $\Gamma = 2$, the grand partition function and correlations can be written in terms of the eigenvalues and eigenvectors of a block Toeplitz matrix. By using the semiperiodic Coulomb potential and taking the continuum limit in the periodic direction so as to have a set of parallel lines as the domain, it is shown that these eigenvalues and eigenvectors can be computed exactly. This allows the pressure and the correlations near a charged wall to be rigorously evaluated. The two-particle correlations obey a sum rule which implies that the state in the vicinity of the wall is a conductor.

KEY WORDS: Block Toeplitz matrices; two-dimensional Coublomb gas; conductors; solvable models.

1. INTRODUCTION AND SUMMARY

The two-dimensional (log-potential) two-component plasma on a lattice, at the special value of the dimensionless coupling $\Gamma = 2$, has been the subject of several recent studies.(1–4) The pioneering work of Gaudin(1) gave expressions for the grand partition function and the correlation functions in terms of a $2M_1M_2 \times 2M_1M_2$ block Toeplitz matrix. The asymptotic behavior of the eigenvalues and eigenvectors of the block Toeplitz matrix was conjectured, which allowed closed-form expressions for the pressure and bulk correlations to be given in the thermodynamic limit. Subsequently, Cornu and Jancovici(2) analyzed there results. Of particular interest to the present study is their observation that by taking the continuum limit in one direction, the closed-form expressions obtained by Gaudin simplify.

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In this paper, both the lattice model and the parallel lines model will be reconsidered, with surface charges along the boundaries in one direction. In Section 2 a semiperiodic potential is used which allows the block Toeplitz matrix to be partially diagonalized, reducing the problem to that of obtaining the eigenvalues and eigenvectors of a $2M_2 \times 2M_2$ block Toeplitz matrix, where M_2 is the number of rows in the nonperiodic direction. For $M_2 = 1$, the pressure and two-particle correlations are given exactly for the lattice model, and it is shown that except for one special choice of the parameters, the system is in a nonconducting phase (for $\Gamma < 2$ the one-dimensional, two-component log-potential Coulomb gas a conducting, while for $\Gamma > 2$ it is insulating; see, e.g., ref. 4) and this phase should persist for general M_2 in the strip system.

In Section 3 the parallel lines limit is considered. An exact calculation of the eigenvalues and eigenvectors of the corresponding $2M_2 \times 2M_2$ block Toeplitz matrix is given, which allows the pressure and correlations near the charged surface to be rigorously calculated in the thermodynamic limit. The correlations near the boundary obey a sum rule which indicates that the phase is now a conductor. The state thus changes from a nonconductor to a conductor as the width of the system is taken to infinity.

2. THE LATTICE MODEL

The system to be studied is a charge neutral and symmetric mixture of positive and negative two-dimensional Coulomb charges. Periodic boundary conditions are applied in the X direction, so that a charge q with complex coordinate $z = x + iy$ and a charge q' with the complex coordinate $z' = x' + iy'$ interact via the potential

$$V(z, z') = -qq' \log\{|\sin \pi(z - z')/L| (L/\pi)\} \tag{2.1}$$

The two-dimensional Coulomb gas is a two-parameter system: (i) the dimensionless coupling

$$\Gamma := q^2/k_B T \tag{2.2}$$

where q is the magnitude of the charges, and (ii) the dimensionless density $\tau\rho$, which is the ratio of the interparticle spacing $1/\rho$ to the "hard-core" diameter τ of the particles. The hard-core or similar regularization of the logarithmic potential is necessary to stop the collapse of positive and negative charge pairs at low temperature.

An alternative to imposing a "hard core" about each of the charges is to divide the domain into a grid of two sublattices, and allow each species to occupy one or the other of the sublattices. This choice of domain, which is necessary for solvability properties, is adopted in this paper.

2.1. Definition of the Model

Consider a rectangle of side lengths L and W in the X and Y directions, respectively. Divide the rectangle into a grid of $M_1 \times M_2$ sites, with lattice points at the coordiniates $(n_1 L/M_1,\ n_2 W/M_2)$, $n_j = 1, 2,..., M_j$ $(j = 1, 2)$. Introduce a second interlacing lattice with coordinates $((n_1 - \phi_1)L/M_1,\ (n_2 - \phi_2)W/M_2)$. Allow positive charges to occupy the first sublattice and negative charges of the same magnitude q to occupy the second. On the boundaries at $y = 0$ and $y = W$ impose surface charges $q\sigma$ and $-q\sigma$, respectively (see Fig. 1). The charges interact via the potential (2.1) and a charge q' at the point (x, y) within the system experiences a potential $\phi(y)$ due to the surface charges where

$$\phi(y) = \pi\sigma q q'(W - 2y) \tag{2.3}$$

The surface charge–surface charge interaction contributes an energy

$$\pi\sigma^2 q^2 WL \tag{2.4}$$

to the system.

The scaled coordinates of the kth positive charge can be written in complex form as

$$w_k = \pi m_k/M_1 + i\pi n_k W/LM_2 \tag{2.5}$$

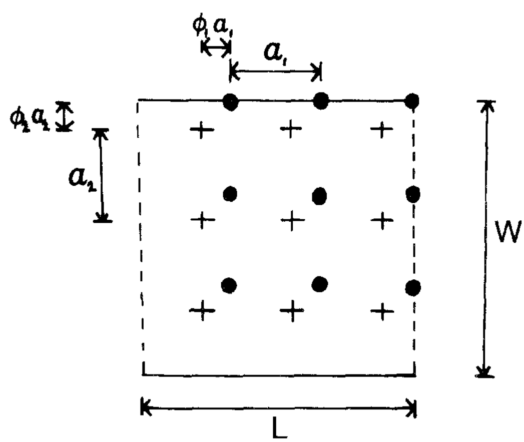

Fig. 1. The lattice geometry. Here $a_1 := L/M_1$ and $a_2 := W/M_2$. The negative charges are restricted to the sublattice denoted by plus signs, while the positive charges are confined to the sublattice denoted by dots. The lower and upper boundaries contain surface charges $q\sigma$ and $-q\sigma$, respectively.

and similarly the coordinates of the kth negative charge can be written as

$$z_k = \pi(m'_k - \phi_1)/M_1 + i\pi(n'_k - \phi_2)W/LM_2 \tag{2.6}$$

where $1 \leqslant n_k,\ n'_k \leqslant M_1$ and $1 \leqslant m_k,\ m'_k \leqslant M_2$. With this notation the Boltzmann factor $W_{N\Gamma}$ for N particles of charge q and N particles of charge $-q$ within the system is

$$W_{N\Gamma} = (\pi/L)^{N\Gamma} \exp\left[2\pi\Gamma\sigma W \sum_{j=1}^{N} (n_j - n'_j + \phi_2)/M_2 - \pi\Gamma\sigma^2 WL\right] \times |F(w_1,..., w_N, z_1,..., z_N)|^{\Gamma} \tag{2.7}$$

where

$$F := \frac{\prod_{1 \leqslant j<k \leqslant N} \sin(w_k - w_j)\sin(z_k - z_j)}{\Pi_{j=1}^{N} \prod_{k=1}^{N} \sin(w_j - z_k)} = (-1)^{N(N-1)/2} \det\left[\frac{1}{\sin(w_j - z_k)}\right]_{j,k=1,...,N} \tag{2.8}$$

and Γ is given by (2.2). The equality in (2.8) follows by substituting $x_j = \exp(2iw_j)$ and $y_j = \exp(2iz_j)$ in the Cauchy double alternant formula.

The partition function $Z_{N\Gamma}(a, b)$, with site variables $a(m, n)$ and $b(m', n')$ introduced for convenience in calculating the correlation functions, is given by

$$Z_{N\Gamma}(a, b) = \sum_{w \in \{r\}} \sum_{z \in \{s\}} \prod_{l=1}^{N} [1 + a(m_l, n_l)][1 + b(m'_l, n'_l)]\, W_{N\Gamma} \tag{2.9}$$

where the sum is over the set of complex numbers $r_{j,k}$ and $s_{j,k}$ taken N at a time,

$$r_{j,k} = \pi j/M_1 + \pi i Wk/LM_2 \tag{2.10}$$

and

$$s_{j,k} = \pi(j - \phi_1)/M_1 + \pi i W(k - \phi_2)/LM_2 \tag{2.11}$$

with $1 \leqslant j \leqslant M_1$ and $1 \leqslant k \leqslant M_2$. The corresponding grand partition function is

$$\Xi_{\Gamma}(a, b) = \sum_{N=0}^{M_1 M_2} \zeta^{2N} Z_{N\Gamma} \tag{2.12}$$

where ζ denotes the fugacity.

2.2. Identities at $\Gamma = 2$

From (2.7) and (2.8) it follows that at $\Gamma = 2$ the Boltzmann factor can be written as

$$W_{N2} = (\pi/L)^{2N} \exp\left[4\pi\sigma W \sum_{j=1}^{N} (n_j - n_j' + \phi_2)/M_2 - 2\pi\sigma^2 WL\right] \times \det\begin{bmatrix} \mathbf{O}_N & \mathbf{A} \\ -(\bar{\mathbf{A}}^T) & \mathbf{O}_N \end{bmatrix} \tag{2.13}$$

where $\mathbf{O}_N$ denote the $N \times N$ zero matrix and

$$\mathbf{A} = \left[\frac{1}{\sin(w_j - z_k)}\right]_{j,k=1,\dots,N} \tag{2.14}$$

If was observed by Gaudin[1] that if the identity (2.13) is substituted in the formulas (2.9) and (2.12) for the grand partition function, then the resulting expression is an expansion in minors of a $2M_1M_2 \times 2M_1M_2$ block matrix. Thus

$$\Xi_2(a, b) = e^{-2\pi\sigma^2 WL} \det\left(\mathbf{1}_{2M_1M_2} + \zeta\begin{bmatrix} \mathbf{O}_{M_1M_2} & \mathbf{A}_1(a) \\ \mathbf{A}_2(b) & \mathbf{O}_{M_1M_2} \end{bmatrix}\right) \tag{2.15}$$

where

$$\mathbf{A}_1(a) = \left[\frac{\pi(1 + a(l_1, l_2))e^{2\pi\sigma W(l_2 - l_2' + \phi_2)/M_2}}{L\sin(r_{l_1,l_2} - s_{l_1',l_2'})}\right] \tag{2.16}$$

and

$$\mathbf{A}_2(b) = \left[\frac{-\pi(1 + b(l_1, l_2))e^{2\pi\sigma W(l_2' - l_2 + \phi_2)/M_2}}{L\sin(\bar{r}_{l_1',l_2'} - \bar{s}_{l_1,l_2})}\right] \tag{2.17}$$

In (2.15), $\mathbf{1}_{2M_1M_2}$ denotes the $2M_1M_2 \times 2M_1M_2$ identity matrix, and in the matrices (2.16) and (2.17) the rows are labeled by the ordered pair (l_1, l_2) while the columns are labeled by (l_1', l_2'), where

$$1 \leqslant l_1, l_1' \leqslant M_1 \quad \text{and} \quad 1 \leqslant l_2, l_2' \leqslant M_2 \tag{2.18}$$

2.3. Zeros of the Grand Partition Function

From (2.15) it follows that

$$\Xi_2(0, 0) = e^{-2\pi\sigma^2 WL} \prod_{s=+,-} \prod_{p=1}^{M_1} \prod_{q=1}^{M_2} [1 + \zeta\lambda(s, p, q)] \tag{2.19}$$

where $\lambda(s, p, q)$ denotes the eigenvalues of the matrix $\mathbf{K}$, which is defined in terms of the matrix (2.16) as

$$\mathbf{K} = \begin{bmatrix} \mathbf{O}_{M_1 M_2} & \mathbf{A}_1(0) \\ -(\overline{\mathbf{A}_1(0)})^T & \mathbf{O}_{M_1 M_2} \end{bmatrix} \tag{2.20}$$

Since $\mathbf{K}$ is anti-Hermitian, the eigenvalues must be pure imaginary and occur in complex conjugate pairs. Thus, from (2.19), the zeros of the grand partition function when regarded as a function of ζ^2 are all on the negative real axis.

2.4. The Correlation Functions

The (dimensionless) density at the point $\boldsymbol{l} = (l_a, l_b)$ on the sublattice containing the positive particles is given by

$$\rho_+(\boldsymbol{l}) = \frac{1}{\Xi_2(0, 0)} \frac{\delta}{\delta a(l_a, l_b)} \Xi_2(a, b)\bigg|_{a=b=0} \tag{2.21}$$

To calculate the functional derivative, we note that each row in the determinant (2.15) contains a different site fugacity. The functional differentiation can thus be performed row by row to give

$$\rho_+(\boldsymbol{l}) = \frac{1}{\Xi_2(0, 0)} \det(\mathbf{1}'_{2M_1 M_2} + \zeta \mathbf{K}) \tag{2.22}$$

where $\mathbf{1}'_{2M_1 M_2}$ denotes the identity matrix except for the diagonal entry in the row (l_a, l_b) of the top half, which is equal to zero. Use of (2.15) and some simple manipulation then gives

$$\rho_+(\boldsymbol{l}) = \zeta \langle +\boldsymbol{l} | \mathbf{K}(\mathbf{1}_{2M_1 M_2} + \zeta \mathbf{K})^{-1} | +\boldsymbol{l} \rangle \tag{2.23}$$

where the notation

$$\langle s\boldsymbol{l} | \mathbf{X} | s'\boldsymbol{l}' \rangle \tag{2.24}$$

denotes the element in row $\boldsymbol{l}$ and column $\boldsymbol{l}'$ of the block ss' $(s, s' = +, -)$ of the matrix $\mathbf{X}$.

Similarly

$$\rho_-(\boldsymbol{l}) = \zeta \langle -\boldsymbol{l} | \mathbf{K}(\mathbf{1}_{2M_1 M_2} + \zeta \mathbf{K})^{-1} | -\boldsymbol{l} \rangle \tag{2.25}$$

and an analogous argument gives that the two-particle correlations can be expressed as

$$\begin{aligned} \rho^T_{s_1 s_2}(\boldsymbol{l}_1, \boldsymbol{l}_2) = & -\zeta^2 \langle s_1 \boldsymbol{l}_1 | \mathbf{K}(\mathbf{1}_{2M_1 M_2} + \zeta \mathbf{K})^{-1} | s_2 \boldsymbol{l}_2 \rangle \\ & \times \langle s_2 \boldsymbol{l}_2 | \mathbf{K}(\mathbf{1}_{2M_1 M_2} + \zeta \mathbf{K})^{-1} | s_1 \boldsymbol{l}_1 \rangle \end{aligned} \tag{2.26}$$

The matrix elements above can be expressed in terms of the eigenvalues and eigenvectors of $\mathbf{K}$. Let $\mathbf{v}(s, p, q)$ denote the normalized eigenvector of $\mathbf{K}$ corresponding to the eigenvalue $\lambda(s, p, q)$. Since $\mathbf{K}$ is anti-Hermitian, eigenvectors corresponding to distinct eigenvalues are orthogonal. If we assume that all the eigenvalues are distinct, the vectors $|s_2 \boldsymbol{l}_2\rangle$ and $\langle s_1 \boldsymbol{l}_1|$ can be Fourier decomposed in terms of the eigenvectors. It is then a simple exercise to show that

$$\langle s_1 \boldsymbol{l}_1 | \mathbf{K}(\mathbf{1}_{2M_1M_2} + \zeta \mathbf{K})^{-1} | s_2 \boldsymbol{l}_2\rangle$$
$$= \sum_{s=+,-} \sum_{p=1}^{M_1} \sum_{q=1}^{M_2} \frac{\lambda(s, p, q)}{1+\zeta\lambda(s, p, q)} \bar{v}(s, p, q; s_1 \boldsymbol{l}_1)\, v(s, p, q; s_2 \boldsymbol{l}_2) \quad (2.27)$$

where $v(s, p, q; s'\boldsymbol{l})$ denotes the element in position $s'\boldsymbol{l}$ (s' denotes the half) of $\mathbf{v}(s, p, q)$.

2.5. Eigenvalues and Eigenvectors of K

From (2.16), (2.10), and (2.11), the elements of the matrix $\mathbf{A}_1(0)$ have the Toeplitz structure $a_{(l_1 - l_1', l_2 - l_2')}$, where

$$a_{(l,l')} = \frac{\pi e^{2\pi\sigma W(l' + \phi_2)/M_2}}{L \sin \pi[(l+\phi_1)/M_1 + iW(l'+\phi_2)/LM_2]} \quad (2.28)$$

Furthermore, they have the antiperiodicity property

$$a_{(l+M_1,l')} = -a_{(l,l')} \quad (2.29)$$

It thus follows that the similarity transformation

$$\begin{bmatrix} \mathbf{U}^{-1} & \mathbf{O}_{M_1M_2} \\ \mathbf{O}_{M_1M_2} & \mathbf{U}^{-1} \end{bmatrix} \mathbf{K} \begin{bmatrix} \mathbf{U} & \mathbf{O}_{M_1M_2} \\ \mathbf{O}_{M_1M_2} & \mathbf{U} \end{bmatrix} \quad (2.30)$$

where

$$\mathbf{U} = \frac{1}{(M_1M_2)^{1/2}} [e^{-2\pi i l_1'(k_1 - 1/2)/M_1} \delta_{l_1', k_2}] \quad (2.31)$$

diagonalizes the blocks of $\mathbf{K}$ labeled by l_1 and l_1'.

Denoting the matrix (2.30) by $\mathbf{T}$, we have

$$\mathbf{T} = \begin{bmatrix} \mathbf{O}_{M_1M_2} & \mathbf{D}(k_1)\delta_{l_1,k_1} \\ -\bar{\mathbf{D}}^T(k_1)\,\delta_{l_1,k_1} & \mathbf{O}_{M_1M_2} \end{bmatrix} \quad (2.32)$$

where

$$\mathbf{D}(k_1) = \left[\sum_{l=1}^{M_1} a_{l,l_2-k_2} e^{2\pi i l(k_1-1/2)/M_1} \right]_{l_2,k_2=1,\dots,M_2}$$
$$:= [d_{l_2-k_2}] \tag{2.33}$$

The identity [ref. 5, Eq. (3.12)]

$$\sum_{l=1}^{M} \frac{e^{2\pi i l(p-1/2)/M}}{\sin \pi(l+\phi)/M} = \frac{M}{\sin \pi\phi} e^{-2\pi i\phi(p-(M+1)/2)/M} \tag{2.34}$$

allows the sum in (2.33) to be evaluated to give

$$d_{l_2-k_2} = \pi M_1 \exp\left(- \frac{2\pi i\phi_1(k_1-(M_1+1)/2}{M_1} \right)$$
$$\times \left(\frac{\exp\{2\pi r(k_1-1/2-M_1/2+\sigma L)(l_2-k_2+\phi_2)/M_1\}}{L \sin \pi[\phi_1 + ir(l_2-k_2+\phi_2)]} \right) \tag{2.35}$$

where

$$r = WM_1/LM_2 = a_2/a_1 \tag{2.36}$$

It follows from (2.30)–(2.32) that the elements of the eigenvectors of **K** have the tensor product structure

$$v(s, p, q; s'\mathbf{l}') = e^{-2\pi i l_1'(p-1/2)/M_1} \otimes u(s, p, q; s'l_2') \tag{2.37}$$

where $u(s, p, q; s'l_2)$ denotes the element $s'l_2$ of the $2M_2 \times 2M_2$ block matrix

$$\begin{bmatrix} \mathbf{O}_{M_2} & \mathbf{D}(p) \\ -\bar{\mathbf{D}}^T(p) & \mathbf{O}_{M_2} \end{bmatrix} \tag{2.38}$$

Furthermore, the eigenvalues $\lambda(s, p, q)$ of **K** are just the eigenvalues of the $2M_2 \times 2M_2$ block Toeplitz matrix (2.38) for each $p = 1, 2,\dots, M_1$.

In general it is not possible to explicitly calculate the eigenvalues and eigenvectors of (2.38). However, the case $M_2 = 1$ is tractable and the thermodynamic limit can be obtained.

2.4. One-Dimensional Sublattices

With $M_2 = 1$, which corresponds to both the sublattice containing the positive charges and the sublattice containing the negative charges being one dimensional, (2.38) is a 2×2 matrix. The eigenvalues $\lambda(s, p)$ and corresponding eigenvectors $u(s, p)$ are thus simple to calculate. Taking $\sigma = 0$, we find

$$\zeta\lambda(s, p) = i\, \mathrm{sgn}(s)\, \xi^{1/2} e^{2\pi r\phi_2(p - 1/2 - M_1/2)/M_1} \tag{2.39}$$

where

$$\xi = (\pi M_1 \zeta/L)^2\, |\sin \pi(\phi_1 + ir\phi_2)|^{-2} \tag{2.40}$$

and

$$u(s, p) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i\, \mathrm{sgn}(s)\, c e^{-2\pi i\phi_1(p - (M_1 + 1)/2)/M_1} \end{bmatrix} \tag{2.41}$$

with

$$c = |\sin \pi(\phi_1 + ir\phi_2)| / \sin \pi(\phi_1 + ir\phi_2) \tag{2.42}$$

From (2.19), the dimensionless pressure

$$\beta P := \lim_{M_1 \to \infty} \frac{1}{M_1} \log \Xi(0, 0) \tag{2.43}$$

is thus given by

$$\beta P = \int_0^1 \log(1 + \xi e^{2\pi r\phi_2(2t - 1)})\, dt \tag{2.44}$$

Note that if $\phi_2 = 0$, which corresponds to the sublattices lying along the same line, the pressure is that of a "hard-core" perfect gas. (This result has been found previously.[5])

Also, from (2.25)–(2.27), these results allow the two-particle correlations to be evaluated as

$$\rho^T_{++}(l_a, l_b) = -\left| \int_0^1 \frac{\xi e^{2\pi i(l_a - l_b)t}}{1 + \xi e^{2\pi r(2t - 1)\phi_2}}\, dt \right|^2 \tag{2.45}$$

and

$$\rho^T_{+-}(l_a, l_b) = \xi \left| \int_0^1 \frac{e^{2\pi i(l_a - l_b - \phi_1)t} e^{\pi r(2t - 1)\phi_2}}{1 + \xi e^{2\pi r(2t - 1)\phi_2}}\, dt \right|^2 \tag{2.46}$$

From these exact expressions, we deduce that the large-separation behavior of the dimensionless charge–charge correlation

$$c^T(l_a, l_b) := 2(\rho^T_{++}(l_a, l_b) - \rho^T_{+-}(l_a, l_b)) \tag{2.47}$$

is

$$c^T(l_a, l_b) \sim \frac{-\alpha(\xi)}{2\pi^2(l_a - l_b)^2} \tag{2.48}$$

where $\alpha(\xi) \leqslant 1$. The behavior (2.48) with $\alpha(\xi) = 1$ is the criterion for a conducting phase.[6] From the above expressions, we find that this criterion holds in the present case if and only if $\phi_1 = 1/2$, $\xi = 1$ and $\phi_2 = 0$. The sublattices are then symmetrically interlaced along the some line.

3. THE LINE MODEL

Suppose that the partition function (2.9) is multiplied by $(L/M_1)^{2N}$ and the limit $M_1 \to \infty$ is taken. The partition function is then that of a system of two sets of interlacing lines of length L parallel to the X axis. The first set starts at the points $y = nW/M_2$ ($n = 1, 2,..., M_2$) and is available to the positive charges, while the second set starts at the points $y = (n - \phi_2) W/M_2$ and is available to the negative charges. The boundaries at $y = 0$ and $y = W$ again have surface charges $q\sigma$ and $-q\sigma$, respectively (see Fig. 2). The formulas for the grand partition function (2.19) and the matrix element (2.27) remain valid in the limit $M_1 \to \infty$ [although the label p of the eigenvectors now runs from $-\infty$ to ∞, as will be seen below; also, the RHS of (2.27) needs to be multiplied by L^{-1}]. The operator $\mathbf{K}$ is now an integral operator acting on vectors of length $2M_2$ depending on a continuous variable t_1, defined by the mapping rule

$$\mathbf{K}(\mathbf{v}(t_1)) = \mathbf{u}(t_2) \tag{3.1}$$

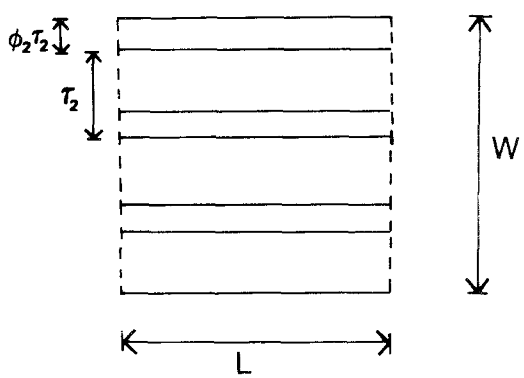

Fig. 2. The parallel line system. The line closest to but distinct from the lower boundary contains negative charges. The lines then alternate between being available to positive and negative charges. The lower and upper boundaries contain surface charges $q\sigma$ and $-q\sigma$, respectively.

where

$$u_{+j}(t_2) = \pi e^{2\pi\sigma\phi_2} \sum_{k=1}^{M_2} e^{2\pi\sigma(j-k)} \int_0^1 \frac{v_{-k}(t_1)}{\sin \pi[t_2 - t_1 + iW(j-k+\phi_2)/LM_2]} dt_1 \tag{3.2}$$

and

$$u_{-j}(t_2) = \pi e^{2\pi\sigma\phi_2} \sum_{k=1}^{M_2} e^{-2\pi\sigma(j-k)} \int_0^1 \frac{v_{+k}(t_1)}{\sin \pi[t_2 - t_1 - iW(j-k-\phi_2)/LM_2]} dt_1 \tag{3.3}$$

[$j = 1, 2, \dots, M_2$; the subscript + (−) refers to the top (bottom) half of the vector].

3.1. The Eigenvalue Problem for the Integral Operator

We seek the solutions of the eigenvalue equation

$$\mathbf{K}(\mathbf{v}(t_1; spq)) = \lambda_{spq} \mathbf{v}(t_2, spq) \tag{3.4}$$

where the indices spq label the eigenvalues. Since from (3.2) and (3.3) the kernel of $\mathbf{K}$ has antiperiod 1 in the continuous difference variable $t_1 - t_2$, we can write

$$\mathbf{v}(t; spq) = e^{-2\pi i t(p-1/2)} \mathbf{x}(sq) \tag{3.5}$$

where p is any integer. The eigenvalue equation (3.4) then becomes the pair of equations

$$\pi e^{2\pi\sigma\phi_2} \sum_{k=1}^{M_2} e^{2\pi\sigma(j-k)} I_{j-k}(p)\, x_{-k}(sq) = \lambda_{spq} x_{+j}(sq) \tag{3.6}$$

$$\pi e^{2\pi\sigma\phi_2} \sum_{k=1}^{M_2} e^{-2\pi\sigma(j-k)} I_{k-j}(p)\, x_{+k}(sq) = \lambda_{spq} x_{-j}(sq) \tag{3.7}$$

where

$$I_l(p) = \int_0^1 \frac{e^{2\pi i(p-1/2)t}}{\sin \pi[t + iW(l+\phi_2)/LM_2]} dt \tag{3.8}$$

The integral (3.7) can be evaluated using the expansion

$$\frac{1}{\sin \pi(t+i\mu)} = -2i \sum_{k=0}^{\infty} e^{\pi i(t+i\mu)(2k+1)}, \qquad \operatorname{Re}(\mu) > 0 \tag{3.9}$$

We find that for $p \geqslant 1$ $(p<1)$

$$I_l(p) = \begin{cases} 2i\,\mathrm{sgn}(p-1)e^{\pi W(l+\phi_2)(2p-1)/LM_2}, & l<0\ (l\geqslant 0) \\ 0, & l\geqslant 0\ (l<0) \end{cases} \tag{3.10}$$

Substituting (3.10) in (3.6) gives, for $p \geqslant 1$, the eigenvalue equations

$$\sum_{k=j+1}^{M_2} e^{-\mu k} Y_k = \gamma e^{-\mu j} X_j \tag{3.11a}$$

$$\sum_{k=1}^{j-1} e^{\mu k} X_k = \gamma e^{\mu j} Y_j \tag{3.11b}$$

where

$$\begin{aligned} \gamma &= -(ie^{-\mu\phi_2}/2\pi)\lambda_{spq}, & \mu &= \pi W(2p-1)/LM_2 + 2\pi\sigma \\ Y_k &= x_{-k}(sq), & X_k &= x_{+k}(sq) \end{aligned} \tag{3.12}$$

while for $p<1$ we obtain

$$\begin{aligned} \sum_{k=1}^{j} e^{-\mu k} Y_k &= -\gamma e^{-\mu j} X_j \\ \sum_{k=j}^{M_2} e^{\mu k} X_k &= -\gamma e^{\mu j} Y_j \end{aligned} \tag{3.13}$$

3.2. Eigenvalue Problem for Some Block Toeplitz Matrices

We note from (3.11) and (3.13) that the eigenvalue problem is that of obtaining the eigenvalues and eigenvectors of the real Hermitian block Toeplitz matrix

$$\begin{bmatrix} \mathbf{O}_{M_2} & \mathbf{B}(p) \\ \mathbf{B}^T(p) & \mathbf{O}_{M_2} \end{bmatrix} \tag{3.14}$$

where for $p \geqslant 1$ (and with $x := e^{-\mu}$)

$$\mathbf{B}(p) = \begin{bmatrix} 0 & x & x^2 & x^3 & \cdots & x^{M_2-1} \\ 0 & 0 & x & x^2 & \cdots & x^{M_2-2} \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & x \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{3.15}$$

while for $p<1$

$$\mathbf{B}(p)=-\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ x^{-1} & 1 & 0 & \cdots & 0 \\ x^{-2} & x^{-1} & 1 & \cdots & 0 \\ \vdots & & & & \\ x^{-M_2+1} & x^{-M_2+2} & x^{-M_2+3} & \cdots & 1 \end{bmatrix} \tag{3.16}$$

To be consistent with (3.11) and (3.13), the first M_2 components of the eigenvectors of these matrices will be denoted by X_j, $j=1,\ldots, M_2$, and the second M_2 components will be denoted by Y_j.

The structure of (3.14) allows M_2 eigenvectors to be found with the property

$$X_k = Y_{M_2+1-k}, \qquad k=1, 2,\ldots, M_2 \tag{3.17}$$

The eigenvalue problem is then that of finding the eigenvalues and eigenvectors of the $M_2 \times M_2$ Hankel matrix obtained by replacing the kth column in (3.15) [or (3.16) if $p<1$] by the $(M+1-k)$th column, $k=1, 2,\ldots, M_2$. The remaining M_2 eigenvectors have the property

$$X_k = -Y_{M_2+1-k}, \qquad k=1, 2,\ldots, M_2 \tag{3.18}$$

and correspond to the negative of the eigenvalue found with (3.17). The eigenvectors and corresponding eigenvalues can thus be labeled by a parameter s, $s=\pm 1$, such that

$$X_k = \operatorname{sgn}(s)\, Y_{M_2+1-k}, \qquad k=1, 2,\ldots, M_2 \tag{3.19}$$

From (2.23) and (2.25), the property (3.19) implies that the density of the positive charges as measured from the charged boundary at $y=0$ is the same as the density of the negative charges at the same distance from the charged boundary at $y=W$, a feature obvious from the Hamiltonian of the system.

The Case $p \geqslant 1$. Let us consider first the case $p \geqslant 1$. From (3.14) and (3.15) we see that there are two zero eigenvalues. For the nonzero eigenvalues, we see from (3.11a) that

$$\gamma x X_{k+1} = \gamma X_k - x Y_{k+1} \tag{3.20}$$

while it follows from (3.11b) that

$$X_k = \gamma x^{-1} Y_{k+1} - \gamma Y_k \tag{3.21}$$

where x is defined between (3.14) and (3.15t). Substituting (3.21) in (3.20) gives the second-order, linear, constant-coefficient difference equation

$$Y_{k+2}+(-x-1/x+x/\gamma^2)\,Y_{k+1}+Y_k=0 \tag{3.22}$$

This is to be solved subject to the boundary condition

$$Y_1=0 \tag{3.23}$$

[choose $j=1$ in (3.11b)].

In addition to (3.22) and (3.23), from (3.21) with $k=1$, (3.19), and (3.19), we have

$$Y_{M_2}=\operatorname{sgn}(s)(\gamma/x)\,Y_2 \tag{3.24}$$

while from (3.21) with $k=M_2-1$, (3.20), and (3.24) we have

$$Y_{M_2}=\frac{x}{1-(x/\gamma)^2}\,Y_{M_2-1} \tag{3.25}$$

These two equations can be used to calculate the eigenvalues.

Solving the difference equation (3.22) with the boundary condition (3.23) in the usual way gives

$$Y_k=\alpha\sin[\theta(k-1)] \tag{3.26}$$

where α is a normalization constant and

$$e^{i\theta}=\tfrac{1}{2}\{x+1/x-x/\gamma^2+[(x+1/x-x/\gamma^2)^2-4]^{1/2}\} \tag{3.27}$$

The eigenvalues γ are thus given in terms of θ by

$$\gamma=\pm\frac{x^{1/2}}{(x+1/x-2\cos\theta)^{1/2}} \tag{3.28}$$

so we must find (M_2-1) distinct values of θ. These values are specified by (3.24)–(3.26), which give the equation

$$\frac{\sin(M_2-1)\theta}{\sin(M_2-2)\theta}\left[1-\left(\frac{\sin\theta}{\sin(M_2-1)\theta}\right)^2\right]=x \tag{3.29}$$

Consideration of the graph of the left-hand side of (3.29) (see Fig. 3) shows that for $0<\theta<\pi$ there are M_2-1 solutions for $x\leqslant x_0$, where

$$x_0=\frac{M_2-1}{M_2-2}\left[1-\left(\frac{1}{M_2-1}\right)^2\right] \tag{3.30}$$

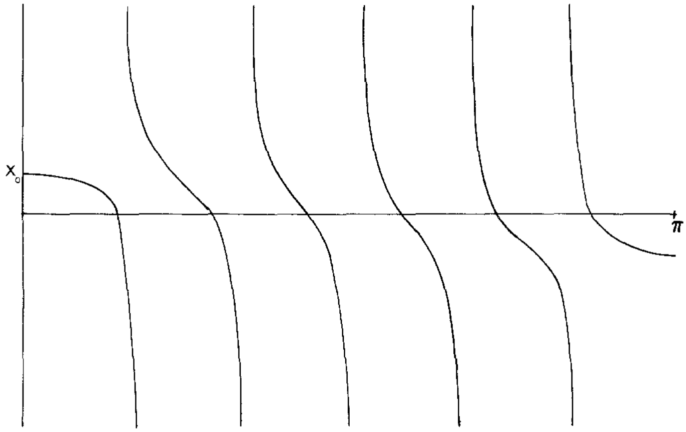

Fig. 3. Graph of the equation

$$x = [\sin(M_2 - 1)\theta/\sin(M_2 - 2)\theta][1 - (\sin\theta/\sin(M_2 - 1)\theta)^2]$$

for $M_2 = 7$ and $0 \leqslant \theta \leqslant \pi$. In general, if $0 \leqslant x \leqslant x_0$, there are $M_2 - 1$ solutions, while if $x > x_0$, there are $M_2 - 2$ solutions.

and $M_2 - 2$ solutions for $x > x_0$. For $x > x_0$, the remaining θ value is given by $\theta = iv$, where

$$\frac{\sinh(M_2 - 1)v}{\sinh(M_2 - 2)v}\left[1 - \left(\frac{\sinh v}{\sinh(M_2 - 1)v}\right)^2\right] = x \tag{3.31}$$

Furthermore, the real solutions $\theta_1, \theta_2, ..., \theta_{M_2 - 1}$ lie in the intervals

$$\frac{\pi(k-1)}{M_2 - 1} < \theta_k < \frac{\pi k}{M_2 - 1}, \qquad k = 1, 2, ..., M_2 - 1 \tag{3.32}$$

and are thus uniformly distributed on the interval 0 to π. From (3.24) and (3.26) the two eigenvalues implies by (3.31) are

$$\gamma = \pm x \frac{\sinh v(M_2 - 1)}{\sinh v} \tag{3.33}$$

Hence, in summary, we have that the eigenvectors $\mathbf{x}(sq)$ in (3.5) have the components

$$X_k = \alpha \sin\theta_q(M_2 - k) = \gamma(x^{-1}Y_{k+1} - Y_k) \tag{3.34a}$$

$$Y_k = \operatorname{sgn}(s)\, \alpha \sin\theta_q(k - 1) \tag{3.34b}$$

where $k = 1, ..., M_2$, $q = 1, ..., M_2 - 1$,

$$x = e^{-\pi W(2p-1)/LM_2 - 2\pi\sigma W/M_2} \tag{3.34c}$$

X_k denotes the components in the top half, Y_k denotes the components in the bottom half, and α denotes the normalization. The allowed values of θ_q are given by (3.29). The corresponding eigenvalues are

$$\lambda_{spq} = 2\pi i x^{-\phi_2} \gamma \tag{3.35a}$$

$$= 2\pi i \,\mathrm{sgn}(s) x^{1-\phi_2} \frac{\sin \theta_q (M_2 - 1)}{\sin \theta_q} \tag{3.35b}$$

$$= \pm \frac{2\pi i x^{1/2 - \phi_2}}{(x + 1/x - 2\cos\theta_q)^{1/2}} \tag{3.35c}$$

where the choice of sign in (3.35c) is determined by (3.35b).

The Case $p<1$. For $p<1$, the problem is to find the eigenvalues and eigenvectors of the matrix (3.14) as specified by (3.16). Comparison of (3.15) and (3.16) shows that the components of the eigenvectors and the eigenvalues are closely related to those found above for $p>1$ and $\gamma \neq 0$.

Explicitly, by making the replacements

$$M_2 \to M_2 + 1, \qquad x \to x^{-1}, \qquad \gamma \to -\gamma x^{-1}, \qquad X_j \leftrightarrow Y_j \tag{3.36}$$

the eigenvalue problem solved for $p>1$ and $\gamma \neq 0$ is mapped to the present problem, provided we first delete the zero entries $(Y_1 = X_{M_2} = 0)$ of the eigenvectors of the former. Thus, from (3.34) the eigenvectors $\mathbf{x}(sq)$ in (3.5) have the components

$$X_k = \alpha \sin \theta_q k \tag{3.37a}$$

$$Y_k = \alpha \,\mathrm{sgn}(s) \sin \theta_q (M_2 - k + 1)$$

$$= \alpha\gamma \,\mathrm{sgn}(s) [\sin \theta_q k - x^{-1} \sin \theta_q (k-1)] \tag{3.37b}$$

where the allowed values of θ_q are given by (3.29), with the replacements (3.36). The formula (3.35c) for the eigenvalues remains unchanged.

3.3. The Thermodynamic Limit

3.3.1. The Strip Free Energy. The free energy per unit length of the strip, f_W, is given in terms of the grand partition function by the formula

$$\beta f_W = (\langle N \rangle / L) \log \zeta - \lim_{L \to \infty} \frac{1}{L} \log \Xi(0, 0)$$

$$:= (\langle N \rangle / L) \log \zeta - \beta P_s$$

where $\langle N \rangle$ is the average number of particles. From (2.19) (with p now ranging from $-\infty$ to ∞) the grand partition function and thus the pressure are given in terms of the eigenvalues of the operator $\mathbf{K}$. The functional dependence on L for the eigenvalues is in the ratio $(2p-1)/L$. Hence, taking the logarithm of the grand partition function gives a Riemann approximation to a definite integral, with p/L becoming a continuous variable, t say. Thus

$$\beta P_s = -2\pi\sigma^2 W + \frac{M_2}{2\pi W} \sum_{q=1}^{M_2}{}^{*} \int_{-\infty}^{\infty} \log\left\{1 + (2\pi\zeta)^2 \frac{e^{-(1-2\phi_2)s}}{2\cosh s - 2\cos\theta_q}\right\} ds \tag{3.38}$$

where the asterisk denotes that for $s > 0$ the sum ranges from $q = 1$ to $M_2 - 1$, $s = 2\pi Wt/M_2$, and θ_q is given by (3.29) for $t > 0$ and by (3.29) with the replacements (3.36) for $t < 0$.

Next we consider the limits

$$W, M_2 \to \infty, \qquad W/M_2 = a_2 \text{ (fixed)} \tag{3.39}$$

so that a two-dimensional domain is obtained. Let us suppose that $\sigma > 0$. Then $x < 1$ for $t > 0$ [recall (3.34c)], $x^{-1} > 1$ for $-\sigma < t < 0$ and $x^{-1} < 1$ for $t < -\sigma$. From (3.30), in the limit $M_2 \to \infty$ the critical value for a complex solution of (3.29) is $x_0 = 1$. Thus for $t > 0$ all solutions of (3.29) are real and similarly for $t < -\sigma$. However, for $-\sigma < t < 0$ there is a complex solution of (3.29) with corresponding eigenvalues

$$\lambda = \pm 2\pi i x^{-\phi_2} \frac{\sinh vM_2}{\sinh v} \tag{3.40}$$

[make the replacements (3.36) in (3.33) and use (3.35a)]. From (3.31), with x replaced by $1/x$, we see that in the limit $M_2 \to \infty$

$$v = \log x \tag{3.41}$$

so for M_2 large the eigenvalue (3.40) behaves as

$$\lambda \sim \pm \pi i x^{-(M_2+\phi_2)}/\sinh v \tag{3.42}$$

Separating this eigenvalue out of (3.38) shows that in the bulk pressure

$$\beta P_B = \lim_{W\to\infty} \frac{1}{W} \beta P_s \tag{3.43}$$

the first term in (3.38), which depends explicitly on the surface charge, cancels exactly the contribution from (3.42).

We have seen that the real values of θ_q are uniformly distributed in the interval 0 to π. Thus, substituting the second term of (3.38) in (3.43) gives

$$\beta P_B = \frac{1}{2(\pi a_2)^2} \int_0^\pi d\theta \int_{-\infty}^{\infty} ds \log \left\{ 1 + (2\pi\zeta)^2 \frac{e^{-s(1-2\phi_2)}}{2(\cosh s - \cos\theta)} \right\} \quad (3.44)$$

With $\phi_2 = 1/2$ and a rescaling of ζ, this result agrees with the expression derived by Cornu and Jancovici[2] [Eq. (3.2)] from Gaudin's[1] conjectured exact result for the bulk pressure of the lattice model of Section 2.1 at $\Gamma = 2$.

The virial expansion for P_B can be obtained by expanding the integrand in (3.44) near $s = 0$ and $\theta = 0$. The resulting integral is essentially that studied by Gaudin[1] [Eqs. (50)–(62)] for the lattice model. Using Gaudin's working, we obtained the leading-order behavior

$$\beta P_B \sim \frac{\rho}{2} - \frac{\rho}{2 \log \rho} + \cdots \quad (3.45)$$

which is identical to that of the lattice model.

3.3.2. Relationship to a Theorem of Widom. From (2.15), and the working of the above sections, the strip pressure can be written in the form

$$\beta P_s = -2\pi\sigma^2 W + \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \det \left(\mathbf{1}_{2M_2} + \zeta \begin{bmatrix} \mathbf{O}_{M_2} & \mathbf{B}(t) \\ \mathbf{B}^T(t) & \mathbf{O}_{M_2} \end{bmatrix} \right) dt \quad (3.46)$$

where

$$x = e^{-2\pi a_2 (t+\sigma)} \quad (3.47)$$

and $\mathbf{B}(t)$ is given by (3.15) for $t > 0$ and by (3.16) for $t < 0$. The sum of the matrices in (3.46) gives a block Toeplitz matrix. In the special case $\sigma = 0$, and assuming $t \neq 0$, the criteria necessary for the validity of a theorem of Widom (ref. 7; see also ref. 8) regarding the asymptotic behavior of the determinant of block Toeplitz matrices can be verified. Application of the theorem in (3.46) then gives the result (3.44). However, for $\sigma \neq 0$ and $-\sigma < t < 0$ we have noted that $x^{-1} > 1$. This causes the criteria for Widom's theorem to be violated (the matrix elements no longer form a convergent Fourier series) and indeed the asymptotic behavior of the determinant [due to the appearance of the complex solution in (3.29)] is no longer that given by Widom's theorem.

3.4. The Correlations near the Charged Wall

The calculation of the correlation functions in the thermodynamic limit requires the large-M_2 behavior of the eigenvectors. For θ_q real (which, as we have seen, is all cases except for a single pair of eigenvalules when $-\sigma < t < 0$), θ_q tends to the continuous variable θ, $0 \leqslant \theta \leqslant 2\pi$. Furthermore, a simple calculation gives that the normalization constant α behaves as

$$\alpha^2 \sim 1/M_2 \tag{3.48}$$

It remains to consider the complex solution $\theta_q = iv$ of (3.29). From (3.37) and (3.41), for large M_2 the corresponding eigenvectors have the components

$$X_k = \hat{\alpha} \sinh(k \log x) \tag{3.49a}$$

$$Y_k = \operatorname{sgn}(s)\, \hat{\alpha} x^{-(M_2+1-k)} \tag{3.49b}$$

where $\hat{\alpha}$ is the normalization constant. The large-M_2 behavior of $\hat{\alpha}$ is easily seen to be

$$\hat{\alpha}^2 \sim x^{2(M_2+1)}(x^{-2} - 1) \tag{3.50}$$

so consequently, for fixed k,

$$X_k \sim 0 \tag{3.51a}$$

$$Y_k \sim \operatorname{sgn}(s)(x^{-2} - 1)^{1/2} x^k \tag{3.51b}$$

Substituting the above asymptotic behaviors into the formula (2.27) for the matrix elements allows the correlations near the charged boundary at $y = 0$ to be calculated. Substituting the evaluation of the matrix elements into (2.23), (2.25), and (2.26), we then find

$$\rho_+(k) = \frac{2\zeta^2}{\pi} \int_{-\infty}^{\infty} dt \int_0^{\pi} d\theta\, \frac{(X_k(t, \theta))^2}{(1/|\lambda|)^2 + \zeta^2} \tag{3.52a}$$

$$\rho_-(k) = \frac{2\zeta^2}{\pi} \int_{-\infty}^{\infty} dt \int_0^{\pi} d\theta\, \frac{(Y_k(t, \theta))^2}{(1/|\lambda|)^2 + \zeta^2} + \int_{-\sigma}^{0} (\hat{Y}_k(t))^2\, dt \tag{3.52b}$$

$$\rho^T_{++}(l_1 - l_2; k_1, k_2) = -\left(\frac{2\zeta^2}{\pi}\right)^2 \left| \int_{-\infty}^{\infty} dt\, e^{-2\pi i(l_1 - l_2)t} \times \int_0^{\pi} d\theta\, \frac{X_{k_1}(t, \theta)\, X_{k_2}(t, \theta)}{(1/|\lambda|)^2 + \zeta^2} \right|^2 \tag{3.52c}$$

$$\rho^T_{--}(l_1-l_2;k_1,k_2) = -\left(\frac{2\zeta^2}{\pi}\right)^2 \left| \int_{-\infty}^{\infty} dt\, e^{-2\pi i(l_1-l_2)t} \right.$$
$$\times \int_0^\pi d\theta \frac{Y_{k_1}(t,\theta)\, Y_{k_2}(t,\theta)}{(1/|\lambda|)^2+\zeta^2}$$
$$\left. + \frac{\pi}{2\zeta^2} \int_{-\sigma}^{0} \hat{Y}_{k_1}(t)\, \hat{Y}_{k_2}(t)\, dt \right|^2 \tag{3.52d}$$

$$\rho^T_{+-}(l_1-l_2;k_1,k_2) = \left(\frac{2\zeta}{\pi}\right)^2 \left| \int_{-\infty}^{\infty} dt\, e^{-2\pi i(l_1-l_2)t} \right.$$
$$\left. \times \int_0^\pi d\theta \frac{(1/|\lambda|)\, X_{k_1}(t,\theta)\, Y_{k_2}(t,\theta)}{(1/|\lambda|)^2+\zeta^2} \right|^2 \tag{3.52e}$$

where

$$X_k(t,\theta) := \begin{cases} |\gamma|\,[x^{-1}\sin\theta k - \sin\theta(k-1)], & t>0 \\ \sin\theta k, & t<0 \end{cases} \tag{3.53a}$$

$$Y_k(t,\theta) := \begin{cases} \sin\theta(k-1), & t>0 \\ |\gamma|\,[\sin\theta k - x^{-1}\sin\theta(k-1)], & t<0 \end{cases} \tag{3.53b}$$

$$\hat{Y}_k(t) := x^k(x^{-2}-1)^{1/2} \tag{3.53c}$$

x is given by (3.47) and γ is given by (3.28).

The integration over θ can be explicitly carried out in all the above expressions. However, our primary remaining task is to determine the phase of the system in the vicinity of the wall and this calculation does not require that the integration over θ be performed until a further simplification is made.

Using (3.52) and (3.53), tabulations of the density profiles are given in Table I for $\sigma = 0$ and $\sigma = 1$.

3.4.1. The Bulk Correlations. In the limit $k \to \infty$, it is easy to show from (3.52) and (3.53) that

$$\rho_+(k) \to \rho_+ \qquad \text{and} \qquad \rho_-(k) \to \rho_- \tag{3.54}$$

where

$$\rho_+ = \rho_- = \frac{1}{2}\rho = \frac{1}{2}\zeta \frac{\partial(\beta P_B)}{\partial\zeta} \tag{3.55}$$

and βP_B is given by (3.44) (this limiting behavior is also evidenced in Table I). Also, the limits $k_1, k_2 \to \infty$ give formulas for the two-particle

Table I. Density Profiles with $\zeta=0.1$, $\Phi_2=0.5$, and Surface Charge σ As Indicated[a]

k	$[\rho_+(k)-\rho_+]/\rho_+$	$[\rho_-(k)-\rho_-]/\rho_-$
$\sigma=0$		
1	−0.012	−0.429
2	0.000046	−0.056
3	0.00045	−0.011
4	0.0002	−0.0025
$\sigma=1$		
1	−0.148	7.65
2	−0.032	0.198
3	−0.0079	0.03
4	−0.002	0.006

[a] Recall that a line containing negative charges is closest to the boundary, which explains the large excess $[\rho_-(1)-\rho]/\rho$.

correlations which only depend on the difference k_1-k_2, as expected. These limiting formulas are equivalent to those presented by Cornu and Jancovici[2] [Eqs. (3.4) and (3.5)].

3.4.2. The Continuum Limit. In the limit

$$a_2, \zeta \to 0, \qquad \zeta/a_2=\mu \text{ (fixed)} \tag{3.56}$$

it has been shown by Cornu and Jancovici[2,3] that the Coulomb gas at $\Gamma=2$ is equivalent to the free Fermi field. The two-particle correlation functions (3.52c)–(3.52e) tend to well-defined quantities in this limit. It can readily be verified that the expressions obtained agree with those calculated by Cornu and Jancovici[3] [Eqs. (3.7)] for the continuous model near a charged wall.

3.5. Sum Rules

It is straightforward to check from (3.53) that the components of the eigenvectors have the orthonormality property

$$\left(\frac{2}{\pi}\right)\sum_{k=1}^{\infty} X_k(t,\theta)\,X_k(t,\phi)=\delta(\theta-\phi) \tag{3.57}$$

and similarly for $Y_k(t,\theta)$. Also,

$$\sum_{k=1}^{\infty} Y_k(t,\theta)\,\hat{Y}_k(t)=0 \tag{3.58}$$

and

$$\sum_{k=1}^{\infty} (\hat{Y}_k(t))^2 = 1 \tag{3.59}$$

The identity (3.57) is a continuous version of the orthonormality property of the original eigenvectors of the matrix (3.14). As noted by Cornu and Jancovici,[2] these properties and the structure of (3.52c)–(3.52e) imply the validity of the compressibility and perfect screening sum rules (see, e.g., ref. 2 for the explicit statement of these sum rules).

Another known sum rule (see, e.g., ref. 9) of relevance to the present system states that the excess charge density near the charged boundary should exactly cancel the surface charge. If the system were continuous, the sum rule would read

$$\int_0^{\infty} [\rho_+(y) - \rho_-(y)]\, dy = -\sigma \tag{3.60}$$

For the present line model at large distances from the wall that excess charge density at any line alternatives between $q\rho/2$ and $-q\rho/2$, so the summation which should replace the integration on the left-hand side of (3.60) is not properly defined. However, we can show that

$$\sum_{k=1}^{\infty} [\rho_+(k) - \rho_-(k)] = -\sigma + c \tag{3.61}$$

where c is independent of σ. The sum of the left-hand side is a particular ordering of the original conditionally convergent summation.

Finally, we will show that for large $|l_1 - l_2|$,

$$\rho_{++}(l_1 - l_2, k_1, k_2) + \rho_{--}(l_1 - l_2, k_1, k_2) - 2\rho_{+-}(l_1 - l_2, k_1, k_2) \sim -\frac{f(k_1, k_2)}{[2\pi(l_1 - l_2)]^2} \tag{3.62}$$

where

$$\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} f(k_{12}, k_2) = 1 \tag{3.63}$$

This is the criterion for a conducting phase for a two-dimensional Coulomb gas in a half-space.[10]

The results (3.62) and (3.63) follow from (3.52c)–()3.52e) by first integrating by parts at the discontinuity at $t = 0$. The integrations over θ can now be performed using contour integration (by integrating around an inifinite rectangle: down the line Re $z = -\pi$ to the real axis, across to π, then up the line Re $z = \pi$) to give

$$\rho^T_{--}(l_1 - l_2, k_1, k_2) \sim -\frac{1}{[2\pi(l_1 - l_2)]^2} [x_0(x_0^{-1} - \alpha)\alpha^{k_1 + k_2 - 2}]^2 \quad (3.64a)$$

$$\rho^T_{+-}(l_1 - l_2, k_1, k_2) \sim \frac{1}{[2\pi(l_1 - l_2)]^2} (\alpha^{-1} - x_0^{-1})(x_0^{-1} - \alpha) x_0^2 \alpha^{2k_1 + 2k_2 - 2} \quad (3.64b)$$

$$\rho^T_{++}(l_1 - l_2, k_1, k_2) \sim -\frac{1}{[2\pi(l_1 - l_2)]^2} [x_0(\alpha^{-1} - x_0^{-1})\alpha^{k_1 + k_2}]^2 \quad (3.64c)$$

where

$$x_0 = e^{-2\pi a_2 \sigma} \quad (3.65)$$

and

$$\alpha = b - (b^2 - 1)^{1/2}, \qquad b = \tfrac{1}{2}[x_0 + x_0^{-1} + (2\pi\zeta x_0)^2] \quad (3.66)$$

Since $|\alpha| < 1$, the summations over k_1 and k_2 are simply geometry series, which combine to give (3.63).

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